Finite volume form factors and correlation functions at finite temperature

Summary of dissertation

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1. Introduction

Finite size effects play an important role in modern statistical physics and quantum field theory. From a statistical point of view, finite size effects are relevant if the system is close to a critical point and the correlation length is comparable to the size of the system. In this case the specific heat and other critical quantities have a scaling behaviour as a response to varying the size L, which is fixed by the critical exponents of the infinite volume system. Interesting phenomena occur in quantum field theories as well. The most prominent example is the Casimir force between two neutral macroscopic bodies in vacuum, which can be determined from the volume dependence of the ground state energy.

The knowledge of the properties of finite volume QFT is of central importance in at least two ways. On one hand, numerical approaches to QFT necessarily presume a finite volume box, and in order to interpret the results correctly a reliable theoretical control of finite size corrections is needed. On the other hand, working in finite volume is not necessarily a disadvantage. On the contrary, the volume dependence of the spectrum can be exploited to obtain (infinite volume) physical quantities like the elastic scattering phase shifts or resonance widths.

In the dissertation I investigate finite size effects in 1+1 dimensional integrable field theories. The main subjects of interest are finite volume form factors (matrix elements of local operators on eigenstates of the finite volume Hamiltonian) and expectation values and correlation functions at finite temperature.

1+1 dimensional integrable models attracted considerable interest over the past thirty years. This interest is motivated by the fact that low dimensional integrable models represent nontrivial interacting field theories that are exactly solvable: numerous physical quantities can be exactly calculated in these models. Phenomena qualitatively known in higher dimensions, such as universality, duality, etc. can be quantitatively investigated in 1+1 dimensions. There is also a second motivation which is of a more practical nature. Several systems of condensed matter physics can be described by 1+1 dimensional effective field theories that in many cases lead to integrable models.

In statistical physics, two dimensional critical systems can be described by conformal field theories (CFT) which are fixed points of the renormalization group flow. In this sense, off-critical theories correspond to perturbations of CFT's by some of their relevant operators. Off-critical models do not possess the infinite dimensional symmetry algebra of the fixed point CFT, in particular scale invariance is lost. However, it was observed by A. B. Zamolodchikov that in certain situations there remains an infinite number of conserved quantities. In this case the resulting theory is still integrable and can be described by a factorized scattering theory. The integrals of motion restrict the possible bound state structure and mass ratios in the theory. Assuming further the bootstrap principle, ie. that all bound states belong to the same set of asymptotic particles, it is possible to construct the S-matrix with only a finite number of physical poles.

Integrability can be exploited to gain information about the off-shell physics as well. Applying the ideas of analytic S-matrix theory and the bootstrap principle to form factors one obtains a rather restrictive set of equations which they have to obey. These equations can be considered as axioms for the form factor bootstrap, and supplied with the principles of maximum analyticity and the cluster property they contain enough information to determine the form factors completely. Once the form factors are known, it is possible to construct correlation functions trough the spectral expansion.

Integrability also offers powerful methods to explore the finite size properties of these models. It is possible to obtain the exact Casimir energy by means of the Thermodynamic Bethe Ansatz (TBA) or the Destri – De Vega nonlinear integral equations (NLIE), both of which can be extended to describe excited state energies as well. However, less is known about finite size corrections to off-shell quantities, like expectation values or correlation functions.

By Euclidean invariance correlation functions in a finite volume correspond to the evaluation of thermal correlations. These objects can be compared directly to experiments, and they attracted quite a lot of interest over the last decade . One possible way to approach finite temperature correlation functions is by establishing an appropriate spectral representation in finite volume. This constitutes the first motivation for our work, because finite volume form factors play a central role in this approach.

Besides being a promising tool to obtain correlation functions, finite volume form factors provide means to verify the bootstrap approach to form factors. The connection between the scattering theory and the Lagrangian (or perturbed CFT) formulation is rather indirect, however, it is generally believed, that the solutions of the bootstrap axioms correspond to the local operators of the field theory. Previous tests of this approach concern quantities which are constructed from various integrals over the form factors, whereas the multi-particle matrix elements themselves have not been accessible. On the other hand, finite volume form factors can be obtained numerically in the perturbed CFT setting by means of the Truncated Conformal Space Approach (TCSA), and establishing the connection to infinite volume form factors provides a way to directly test the solutions of the form factor bootstrap. This constitutes the second motivation for our work.

2. Contents of the dissertation

In the first chapter of the thesis I give a short introduction into 1+1 dimensional integrable QFT and its connection with Conformal Field Theory. Chapters 2, 3 and 4 are devoted to the study of finite volume form factors in diagonal scattering theories, while chapter 5 deals with vacuum expectation values and correlation functions at finite temperature.

The methods used troughout this work are common in the field of integrable models. They include investigations of correlation functions in the form of spectral representations, applications of the form factor axioms, and form factor perturbation theory. The main numerical tool is the Truncated Conformal Space Approach (TCSA), which is built on the knowledge of the CFT governing the UV behaviour of the massive integrable model. This method uses information coming from an exactly solved strongly interacting system (the theory at the critical point), as opposed to standard perturbation theory or momentum-space renormalization group methods, which only work in the vicinity of a free theory.

In chapter 2 I investigate finite volume form factors using analytic methods. I derive an expression for the elementary form factors (matrix elements between a multiparticle state and the vacuum) by comparing the spectral representation of correlation functions established in finite and infinite volume. Based on this rigorous result I conjecture a formula for arbitrary form factors without disconnected pieces. Diagonal matrix elements require special care because of the presence of various disconnected terms and the ambiguity of diagonal matrix elements in infinite volume. Upon inspection of the form factor axioms (Watson's equations) I compare the symmetric and connected evaluation schemes for infinite volume diagonal form factors proposed earlier in the literature. Form factor perturbation theory is used to derive diagonal finite volume form factors of one-particle and two-particle states. I also conjecture the general result.

Following the analytic study, in chapter 3 I turn to numerical methods. I investigate the massive Lee-Yang model and the critical Ising model in a magnetic field and consider form factors of the perturbing field, and the energy operator, respectively. TCSA is used as a numerical tool to obtain finite volume spectra and form factors. First I identify finite volume states at different values of the volume L: the numerical data is matched with predictions of the Bethe-Yang equations. I then calculate the finite volume form factors and compare them to the analytic formulas (the infinite volume form factors were already available in both models). This way it is possible directly access the form factor functions along certain one-dimensional sections of the rapidity-space parameterized by the volume L. The choice of the section corresponds to which multi-particle states we pick from the finite volume spectrum and it is only limited by the increasing numerical inaccuracy for higher lying states.

In chapter 4 I develop a framework for evaluating the so-called μ-term, a leading exponential correction to energies of multi-particle scattering states (as calculated from the asymptotic Bethe Ansatz) and finite volume form factors. The calculations are based on a simple quantum mechanical picture of bound state quantization in finite volume.

In the second part of the work (chapter 5) I develop a method to evaluate correlation functions at finite temperature: finite volume is introduced as a regulator of the otherwise ill-defined Boltzmann sum. I develop a systematic low-temperature expansion using finite volume form factors and show that the individual terms of this series have a well defined $L \to \infty$ limit. In fact, they can be transformed into integral expressions over the infinite volume form factors.

3. The results of the work

- Chapter 2

  1. I show that the matrix elements in finite volume essentially coincide with the infinite volume form factors up to a nontrivial normalization factor, which is related to the particle densities of the finite volume spectrum (eq. 2.16). I show, that this result captures all finite size corrections which scale as powers of $1/L$. I also show that the residual finite size effects are of order $e^{-\mu L}$ with μ a universal characteristic mass scale of the theory,

which does not depend on the particular form factor in question.

2. I conjecture a formula for generic matrix elements without disconnected pieces (eq. 2.18).

3. I derive a general relation between the symmetric and the connected evaluations of the amibigous diagonal form factors in infinite volume (eqs. 2.41 and 2.42, and Theorem 1).

4. I conjecture a general formula for diagonal finite volume form factors, which I express both with the symmetric and connected evaluations of the infinite volume diagonal form factors (eqs. 2.36 and 2.47; in the two simplest cases I also provide a rigorous proof, see eqs. 2.32 and 2.33). I use the expression in terms of the connected evaluations to prove that this result coincides with a conjecture made independently by H. Saleur (Theorem 2).

5. I also conjecture a formula for the case of states with a zero-momentum particle (eq. 2.53). The most general result (eq. 2.54) concerns situations with possibly more than one zero-momentum particles present in both states; this only occurs in theories with multiple particle species.

- Chapter 3

  1. I numerically determine finite volume spectra and form factors by means of the Truncated Conformal Space approach in the scaling Lee-Yang model and the scaling Ising model in a magnetic field. These data are then compared to the predictions of the previous chapter. In all cases (including form factors with disconnected pieces) I observe a satisfactory agreement in the scaling region, where both the residual finite size effects and the truncation errors are negligible.

  2. In the case of the Ising model I observe that in some situations there is no scaling region, ie. the residual finite size corrections remain relevant even in large volumes, which are out of the reach of our TCSA routines. The large exponential corrections are explained by the presence of a μ-term with a small exponent.

- Chapter 4

  1. I derive the μ-term associated to multi-particle energies obtained by the asymptotic Bethe Ansatz (eqs. 4.12 and 4.27) and to finite volume form factors (eq. 4.34).
  2. I confirm the calculations by comparing the predictions to TCSA data.
  3. I show, that in some circumstances it is possible for the constituents of a bound state to unbind. This phenomenon is demonstrated numerically in case of the Ising model, where moving A_3 states with odd momentum quantum numbers dissociate in small volume into conventional A_1A_1 scattering states.

- Chapter 5

  1. I introduce finite volume as a regulator of the otherwise ill-defined Boltzmann sum and establish the spectral representation of n-point functions in terms of the finite volume form factors. I show that it is possible to derive a systematic low temperature expansion with a well-defined $L \to \infty$ limit.
  2. In the case of the one-point function I calculate the first three nontrivial terms (eq. 5.28) and find complete agreement with the LeClair-Mussardo approach. It is pointed out, that the Delfino proposal differs from the LeClair-Mussardo formula exactly at this order.
  3. In the case of the two-point function I calculate the first non-trivial term (eq. 5.35), which is an unpublished result of this work. This leading-order calculation shows agreement with the LeClair-Mussardo approach.

4. Conclusions

In this thesis it was shown, how to derive finite volume form factors in integrable models, given that the S-matrix is known and the infinite volume form factors have already been calculated.

Form factor functions in infinite volume are obtained by solving the form factor axioms of the bootstrap program and then selecting the solutions with the desired symmetry and scaling properties. Although the identification of scattering theories as perturbed conformal field theories is well-established, there were only a few direct tests of the individual form factor functions prior to this work; the usual tests in the literature proceed through evaluation of correlation functions, sum-rules, etc. In this work, on the other hand, form factor functions at different values of the rapidity parameters were directly compared to TCSA data, thus providing evidence for the applicability of the bootstrap approach to form factors.

As an application of finite volume form factors I derived an evaluation scheme for thermal expectation values and correlation functions at finite temperature. This method is built on first principles and it provides a well-defined way to deal with the infinities, which are introduced in any spectral representation by disconnected terms of the form factors and by different contributions to the partition function itself. These infinities cancel as expected, however a well-defined regularization procedure is needed to obtain the left-over finite pieces. This point was missed in the proposals prior to our work. I demonstrated, that the LeClair-Mussardo formula for vacuum expectation values is correct to the third nontrivial order, however a general proof of the validity of the series is still not available.

There are several directions to extend the results presented in this thesis.

- The generalization to non-diagonal scattering theories would be desirable, since these models (for example the $O(3)$ σ-model) serve as effective field theories describing long-range interactions in real-word condensed matter systems (Heisenberg spin chains, etc.). It is expected that the principles laid out in this work carry over to the non-diagonal case, which only poses technical difficulties.

- It is not clear whether some of these results can be extended to massless scattering theories, which can be used to study renormalization group flows with a nontrivial conformal IR fixed point. In the absence of a mass-gap one does not have control over residual finite size effects, therefore some of the derivations presented here do not apply directly to the massless case.

- It would be interesting to derive an exact description of finite volume form factors; a TBA-like integral equation similar to the one describing excited state energies would be desirable. Exact finite volume form factors could then be used to derive an alternative spectral representation for correlation functions with the role of space and time exchanged. This procedure might prove to be more effective than the one presented in the dissertation.

- It would be desirable to investigate the higher order terms of the LeClair-Mussardo proposal for thermal correlations. One possibility is to proceed along the lines laid out in this work and to evaluate the low-temperature expansion in terms of the finite volume form factors. This way one could resolve the ambiguities present in the LeClair-Mussardo formula due to the kinematic poles of the infinite volume form factors.

  Note added: During the completion of the dissertation there appeared a new paper (F.H.L. Essler and R.M. Konik, *Finite Temperature Dynamical Correlations in Massive Integrable Quantum Field Theories*) which deals with the spin-spin correlation functions in the $O(3)-\sigma$ model. The authors used the methods presented in this thesis together with a different evaluation scheme to calculate the low-temperature expansion up to the third order.

Finally I would like to remark that the arguments which led to the results on finite volume form factors are not confined to integrable models. In fact, they can be applied to non-integrable models (including higher dimensional theories) whenever there is a suitable control over the finite size spectrum. For a generic theory this is the case for one-particle and two-particle states below the inelastic threshold.

5. Publications

- Balázs Pozsgay and Gábor Takács: *Form factors in finite volume I: form factor bootstrap and truncated conformal space* Nucl. Phys. **B788** (2007) 167-208, `arxiv:0706.1445[hep-th]`

- Balázs Pozsgay and Gábor Takács: *Form factors in finite volume II: disconnected terms and finite temperature correlators* Nucl. Phys. **B788** (2007) 209-251, `arxiv:0706.3605[hep-th]`

- Balázs Pozsgay: *Lüscher's mu-term and finite volume bootstrap principle for scattering states and form factors* Nucl.Phys. **B802** (2008) 435-457, `arxiv:0803.4445[hep-th]`